Week 14

April 12, 2013

2020B Adv. Cal II

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A <u>parametric surface</u> is a continuous map from region $D \subset \mathbb{R}^2$ to $\mathbb{R}^3$. It is <u>regular</u> if $\vec{r}: D \to \mathbb{R}^3$, $\vec{r}_u$ and $\vec{r}_v$ satisfy $|\vec{r}_u \times \vec{r}_v| > 0$ in the interior of D (equivalently, $\vec{r}_u$ and $\vec{r}_v$ are linearly independent).

Write

$$\vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))$$

$$\overset{\text{or}}{=} x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}.$$

A <u>surface</u> S is a subset in $\mathbb{R}^3$ that admits a regular parametrization.

eg. 1 The sphere $x^2+y^2+z^2=a^2$.

$$\vec{r}(\theta,\varphi) = (a\sin\varphi\cos\theta, a\sin\varphi\sin\theta, r\cos\varphi)$$

$$\vec{r}_\varphi = (r\cos\varphi\cos\theta, r\cos\varphi\sin\theta, -r\sin\varphi)$$

$$\vec{r}_\theta = (-r\sin\varphi\sin\theta, r\sin\varphi\cos\theta, 0)$$

$$|\vec{r}_\varphi \times \vec{r}_\theta| = r^2\sin\varphi > 0 \quad (0,\pi)\times[0,2\pi]$$

This parametrization is regular away from the 2 poles.

eg. 2 The cylinder $\{(x,y,z): x^2+y^2=r^2\}$

$$\vec{r}(\theta,z) = (r\cos\theta, r\sin\theta, z)$$

$$\vec{r}_\theta = (-r\sin\theta, r\cos\theta, 0)$$

$$\vec{r}_z = (0,0,1)$$

$|\vec{r}_\theta \times \vec{r}_z| = r > 0$ regular everywhere.

e.g.3 Let S be a graph of f:

$$\{(x,y,z) : (x,y) \in D,\ z = f(x,y)\}$$

use
$$\vec{r}(x,y) = (x, y, f(x,y))$$
$$\vec{r}_x = (1, 0, f_x),\quad \vec{r}_y = (0, 1, f_y)$$
$$|\vec{r}_x \times \vec{r}_y| = (1 + f_x^2 + f_y^2)^{\frac{1}{2}} > 0, \text{ regular everywhere.}$$

e.g.4 Surfaces of revolution.

Let $(x(t), z(t))$ be a regular curve in the right xz-plane.

Rotate it around the z-axis to obtain S.

$$\vec{r}(\theta, t) = (x(t)\cos\theta,\ x(t)\sin\theta,\ z(t))$$
$$\vec{r}_\theta = (-x\sin\theta,\ x\cos\theta,\ 0)$$
$$\vec{r}_t = (x'\cos\theta,\ x'\sin\theta,\ z')$$
$$|\vec{r}_\theta \times \vec{r}_t| = x(t)\sqrt{x'^2 + y'^2} > 0 \quad (\text{whenever } x(t) > 0).$$

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Given a fcn f on a surface S, we define its integration over S to be

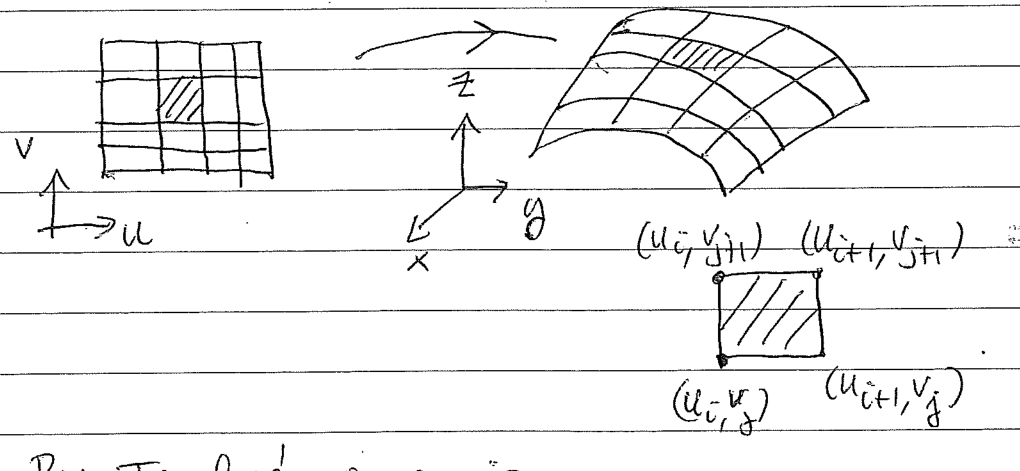
$$\iint_S f(x,y,z)\,d\sigma = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA(u,v)$$

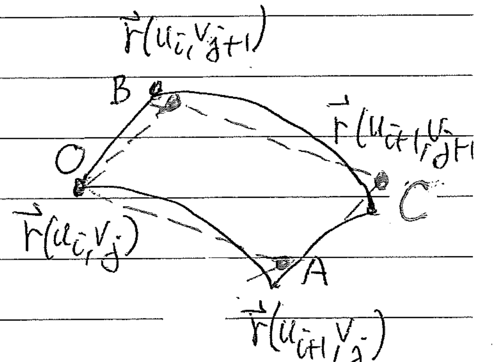
so that

- when $f \equiv 1$, $\iint_S d\sigma = \iint_D |\vec{r}_u \times \vec{r}_v|\,dA(u,v)$ gives the surface area of S,

- when $f \geq 0$, $\iint_S f\,d\sigma$ give the mass of the thin object S with density function f.

Motivation. Let D be a rectangle for simplicity. A partition P in D introduces a corresponding "partition" on S, that is, breaking it up into many small pieces of surfaces.





By Taylor's expansion,

A $(\vec{r}(u_i, v_j) + \vec{r}_u(u_i, v_j)\Delta$

B $(\vec{r}(u_i, v_j) + \vec{r}_v(u_i, v_j)\Delta v$

$$\vec{r}(u_{i+1}, v_j) = \vec{r}(u_i, v_j) + \vec{r}_u(u_i, v_j)\Delta u + \text{higher order terms}$$

$$\vec{r}(u_i, v_{j+1}) = \vec{r}(u_i, v_j) + \vec{r}_v(u_i, v_j)\Delta v + \text{higher order terms}$$

Ignoring the higher order terms, the small piece is approximated by the parallogram at $OABC$. Then the area of the small piece is approximately equal to the area of the parallelogram.

whose area is $\left| \vec{r}_u \times \vec{r}_v \right| (u_i, v_j)\, \Delta u_i \Delta v_j$.

$\therefore$ The "Riemann sum" of the surface integral is

$$\sum_{i,j} f(\text{tag pts}) \left| \vec{r}_u \times \vec{r}_v \right| \Delta u_i \Delta v_j$$

Let $\|P\| \to 0$, get

$$\iint_D f(\vec{r}(u,v)) \left| \vec{r}_u \times \vec{r}_v \right| (u,v)\, dA(u,v).$$

e.g. 1. Find the surface area of the sphere

$$x^2 + y^2 + z^2 = a^2$$

A standard parametrization of the sphere is:

$$(\varphi, \theta) \longmapsto a \sin\varphi \cos\theta\, \hat{i} + a \sin\varphi \sin\theta\, \hat{j} + a \cos\varphi\, \hat{k}$$

$$\vec{r}_\varphi \times \vec{r}_\theta = a^2 \sin^2\varphi \cos\theta\, \hat{i} + a^2 \sin^2\varphi \cos\theta\, \hat{j} + a^2 \sin\varphi \cos\varphi\, \hat{k}$$

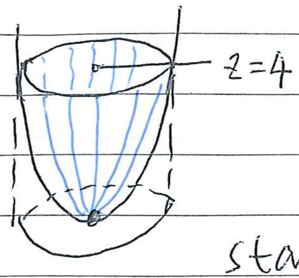
$$\left| \vec{r}_\varphi \times \vec{r}_\theta \right| = a^2 \sin\varphi.$$

$$\therefore \text{ surface area} = \iint_{[0,\pi]\times[0,2\pi]} \left| \vec{r}_\varphi \times \vec{r}_\theta \right| dA(\varphi, \theta)$$

$$= a^2 \int_0^{2\pi} \int_0^{\pi} \sin\varphi \, d\varphi\, d\theta$$

$$= 4\pi a^2. \quad \#$$

e.g. 2 The surface S is cut from $z = x^2+y^2$, at $z=0, 4$.
Find its surface area.



$z=4$ and
$z = x^2+y^2$ cut at $x^2+y^2=4$, ie, D_2.

standard parametrizatin for a graph :

$$\underset{D_2}{(x,y)} \longmapsto (x, y, x^2+y^2)$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1+f_x^2+f_y^2} = \sqrt{1+4(x^2+y^2)}$$

$\therefore$ surface area $$= \iint_{D_2} \sqrt{1+4(x^2+y^2)}\, dA(x,y)$$

$$= \int_0^{2\pi}\int_0^{2} \sqrt{1+4r^2}\, r\,dr\,d\theta \quad \text{(polar coor.)}$$

$$\vdots$$

$$= \frac{\pi}{6}\left(17^{3/2}-1\right) \quad \#$$

e.g. 3 Rotate the curve $x = \cos z$, $z \in [-\pi/2, \pi/2]$, around the z-axis to get S. Find its surface area.

The general formula for S is

$$(t, \theta) \longmapsto (x(t)\cos\theta, x(t)\sin\theta, z(t))$$

Here, it becomes

$$\underset{[-\pi/2,\, \pi/2]\times[0,\, 2\pi]}{(z, \theta)} \longmapsto (\cos z \cos\theta, \cos z \sin\theta, z)$$

$$\vec{r}_z = -\sin z \cos\theta\, \hat{i} - \sin z \sin\theta\, \hat{j} + \hat{k}$$
$$\vec{r}_\theta = -\cos z \sin\theta\, \hat{i} + \cos z \cos\theta\, \hat{j} + 0\, \hat{k}.$$

$$\vec{r}_z \times \vec{r}_\alpha = -\cos\theta\cos z\,\hat{i} - \sin\theta\cos z\,\hat{j} - \cos z\sin z\,\hat{k}$$

$$|\vec{r}_z \times \vec{r}_\alpha| = \cos z\sqrt{1+\sin^2 z}$$

$$\therefore \text{Surface area} = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \cos z\sqrt{1+\sin^2 z}\,dz\,d\alpha$$

$$= 2\pi\int_{-\pi/2}^{\pi/2} \cos z\sqrt{1+\sin^2 z}\,dz$$

$$= 4\pi\int_0^{\pi/2} \cos z\sqrt{1+\sin^2 z}\,dz$$

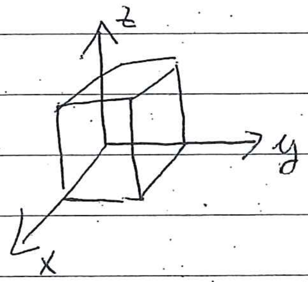
$$= 4\pi\int_0^1 \sqrt{1+t^2}\,dt \qquad \text{(look up Table)}$$

$$\vdots$$

$$= 2\pi\left[\sqrt{2}+\ln(1+\sqrt{2})\right]. \quad \#$$

e.g. 4 Let C be the unit cube in the octant with one vertex at the origin. Find

$$\iint_C xyz\,d\sigma.$$



C is composed of 6 faces.
The faces at $x=0$, $y=0$, $z=0$, have no contribution to the integral as $F(x,y,z) = xyz = 0$ there. So

$$\iint_C xyz\,d\sigma = \iint_{\substack{\text{face}\\ \text{at } x=1}} xyz\,d\sigma + \iint_{\substack{\text{face}\\ \text{at } y=1}} xyz\,d\sigma + \iint_{\substack{\text{face}\\ \text{at } z=1}} xyz\,d\sigma$$

$$= 3\iint_{\substack{\text{face}\\ \text{at } z=1}} xyz\,d\sigma \quad \text{(by symmetry)}$$

The face at $z=1$ has a parametrization

$$(x,y) \longmapsto (x,y,1)$$
$$[0,1]\times[0,1]$$

It is a graph $f(x,y) \equiv 1$, so

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1+f_x^2+f_y^2} = 1$$

$$\therefore \iint_{\substack{\text{face}\\ \text{at } z=1}} xy\,1\times 1\, dA(x,y) = \int_0^1\int_0^1 xy\, dy\, dx = \frac{1}{4}$$

$$\therefore \iint_C xyz\, d\sigma = 3/4 \quad \#$$

Surface Integrals of Vector fields.

Let $\vec{r}(u,v)$ be a parametrization of a surface S. The vector

$$\vec{r}_u \times \vec{r}_v$$

satisfies $\vec{r}_u\times\vec{r}_v \cdot \vec{r}_u = 0$, $\vec{r}_u\times\vec{r}_v\cdot\vec{r}_v = 0$, hence points in the normal direction. We define the unit vector

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

to be the unit normal vector field of the parametrization $\vec{r}$. It is clear that all parametrizations of S can be divided into 2 classes, whose unit normal vector fields pointing in opposite direction.

A surface with a chosen unit normal vector field is called an <u>oriented surface</u>.

Just like we integrate a vector field along an oriented curve, we are going to define the surface integral of a v.f. over an oriented surface.

Let $\vec{F}$ be a v.f. defined on the surface S and let $\vec{r}: D \to S$ be an admissible parametrization so that the normal vector $\hat{n}$ is given by

$$\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}.$$

We define the surface integral of $\vec{F}$ over S to be

$$\iint_S \vec{F} \cdot \hat{n} \, d\sigma.$$

Using

$$\iint_S \vec{F} \cdot \hat{n} \, d\sigma = \iint_D \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} |\vec{r}_u \times \vec{r}_v| \, dA(u,v)$$

$$= \iint_D \vec{F}(\vec{r}(u,v)) \cdot \vec{r}_u \times \vec{r}_v \, dA(u,v),$$

which is the formula to evaluating the integral.

$\vec{F} \cdot \hat{n}$ is the projection of $\vec{F}$ onto the direction $\hat{n}$.

Hence

$$\iint_S \vec{F} \cdot \hat{n} \, d\sigma$$

give the <u>flux</u> of $\vec{F}$ across S in the direction of $\hat{n}$.

e.g 5. Let S be the graph of $y = x^2$ over $[0,1]\times[0,4]$ in the xz-plane where normal is chosen to pointing to the $-y$-direction. Evaluate

$$\iint_S (yz\hat{i} + x\hat{j} - z^2\hat{k})\cdot\hat{n}\, d\sigma$$

Using x, z as parameters

$$(x,z) \longmapsto (x, x^2, z)$$
$$[0,1]\times[0,4]$$

$$\vec{r}_x = (1, 2x, 0) = \hat{i} + 2x\hat{j} + 0\hat{k}$$
$$\vec{r}_z = (0, 0, 1) = 0\hat{i} + 0\hat{j} + \hat{k}$$

$\vec{r}_x\times\vec{r}_z = 2x\hat{i} - \hat{j} + 0\hat{k} = (2x, -1, 0)$ the y-component $-1 < 0$

$\therefore$ $\vec{r}_x\times\vec{r}_z$ points in the $-y$-direction and

$$\hat{n} = \frac{2x\hat{i} - \hat{j} + 0\hat{k}}{|\vec{r}_x\times\vec{r}_z|}$$

$$\iint_S (yz\hat{i} + x\hat{j} - z^2\hat{k})\cdot\hat{n}\, d\sigma = \iint_D (x^2z\hat{i} + x\hat{j} - z^2\hat{k})\cdot\vec{r}_x\times\vec{r}_z\, dA(x,z)$$

$$= \iint_D (x^2z\hat{i} + x\hat{j} - z^2\hat{k})\cdot(2x, -1, 0)\, dA(x,z)$$

$$= \iint_D (2x^3z - x)\, dA(x,z)$$

$$= \int_0^1\int_0^4 (2x^3z - x)\, dz\, dx$$

$$= 2 \ \#$$